PERIOD RELATIONS, JACOBI FORMS AND EICHLER INTEGRAL

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ABSTRACT. We study period relations of Jacobi forms. It turns out that the relations satisfied by Mordell integral coming from Lerch or Appell sums are the special case of those. The existence of Jacobi integral associated to given period function using generalized Poincaré series is claimed.

July 29, 2010

1. Introduction

It is shown, when Zwegers studied Ramanujan Mock theta functions, the Mordell integral[20],

$$h(\tau, z) := \int_{\mathbb{R}} \frac{e^{\pi i \tau x^2 - 2\pi x z}}{\cosh \pi x} dx$$

satisfies the following relations:

$$-\frac{e^{\pi i z^2/\tau}}{\sqrt{-i\tau}} h(-\frac{1}{\tau}, \frac{z}{\tau}) + h(\tau, z) = 0,$$

$$h(\tau, z) = e^{\frac{\pi i}{4}} h(\tau + 1, z) + e^{-\frac{\pi i}{4}} \frac{e^{\frac{\pi i z^2}{\tau+1}}}{\sqrt{\tau+1}} h(\frac{\tau}{\tau+1}, \frac{z}{\tau+1}),$$

$$h(\tau, z) + h(\tau, z+1) = \frac{2}{\sqrt{-i\tau}} e^{\pi i \frac{(z+\frac{1}{2})^2}{\tau}},$$

and

$$h(\tau, z) + e^{-2\pi i z - \pi i \tau} h(\tau, z + \tau) = 2 e^{-\pi i z - \pi i \frac{\tau}{4}}.$$

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It turns out that these are the part of period relations associated to Jacobi forms, namely, any period function $P(\tau, z)$ of Jacobi integral of weight k and index m (with trivial multiplier system) satisfies

$$P(\tau, z) + \tau^{-k} e^{-2\pi i m \frac{z^2}{\tau}} P(-\frac{1}{\tau}, \frac{z}{\tau}) = 0,$$

$$P(\tau, z) + (\frac{\tau-1}{\tau})^{-k} e^{-2\pi i m \frac{z^2}{\tau}} P(\frac{\tau-1}{\tau}, \frac{z}{\tau}) + (\frac{-1}{\tau-1})^{-k} e^{-2\pi i m \frac{z^2}{\tau-1}} P(\frac{-1}{\tau-1}, \frac{z}{\tau-1}) = 0.$$

From the recent work by Zwegers[28], Bringmann-Ono[1] it turns out that the mock theta functions, which were studied by Ramanujan in his letter[24], are holomorphic parts of weak Maass forms. Based on the modular behavior of mock theta functions Zagier[27] further defined a concept of mock modular forms. However mock modular form can be considered as a special case of modular integral with period.

The concept of modular integral already was introduced by Eichler and studied further by many researchers(see, for instance, [14, 15, 16]). It is well known that Eichler integral plays a role to understand periods of modular forms, which are related to the modular symbols and special values of L-functions(see [13]). Note that a connection between period and Maass wave forms was explored by Lewis and Zagier[17] and further applications have been explored by many researchers[3, 11, 18, 19, 21] since then.

The purpose of this article is to study period relations by introducing a concept of Jacobi integral. In particular we introduce a concept of mock Jacobi form, which was already appeared in several places(see [2, 28]), that is a holomorphic Jacobi integral with a "dual" (true) Jacobi form[8]. It turns out that Lerch sums studied in [28] and Appell functions studied in [25] can be viewed as typical examples of mock Jacobi forms.

This paper is organized as follows. We introduce some useful notations in section 2. In section 3, the concept of Jacobi integral with period functions has

been introduced and a lifting map from Jacobi integrals to Jacobi forms are studied. Examples from the indefinite theta series, Appell function and Jacobi Eisenstein series of weight 2 are introduced. In section 4, period relations, using the relations of Jacobi group, are derived and it is also explained in terms of the parabolic cohomology in the sense of Eichler cohomology[14]. A family of Jacobi integral with theta decomposition was introduced.

In section5, using a generalized Jacobi Poincaré series the existence of Jacobi integral, which may have poles, was claimed. Here we modify the idea by Knopp [14], that is, to introduce a generalized Poincaré series to study Eichler cohomology. The detailed proof goes to Appendix in the final section. In section 6, we study a "mock Jacobi form" and period relations of a family of mock Jacobi forms. Section 7 gives a conclusion of this paper.

2. Definitions and Notations

Let us set up the following notations. Let $\mathcal{H}$ be the usual complex upper half plane and $\tau \in \mathcal{H}, z \in \mathbb{C}^j, j \geq 1$. $\Gamma := \Gamma(1) := SL(2, \mathbb{Z})$. The Jacobi group Γ^J is defined as follows:

Definition 2.1. Let

$$\Gamma^J := \Gamma \propto \mathbb{Z}^{2j} = \{[M, (\lambda, \mu)] | M \in \Gamma, \lambda, \mu \in \mathbb{Z}^j\}.$$

This set Γ^J forms a group under a group law

$$[M_1, (\lambda_1, \mu_1)][M_2, (\lambda_2, \mu_2)] = [M_1 M_2, (\lambda', \mu') + (\lambda_2, \mu_2)],$$

where $\binom{\lambda'}{\mu'} = M_2^t \binom{\lambda_1}{\mu_1}$ and is called the **Jacobi group**. Note that the Jacobi group Γ^J acts on $\mathcal{H} \times \mathbb{C}^j$ as, for each $\gamma = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)] \in \Gamma^J, \lambda, \mu \in \mathbb{Z}^j$,

$$\gamma(\tau, z) = (\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}).$$

Furthermore, for $\gamma = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)] \in \Gamma^J, k \in \frac{1}{2}\mathbb{Z}$ and $\mathcal{M}^{(j)} \in M_{j\times j}(\frac{1}{2}\mathbb{Z})$, let

$$j_{k,\mathcal{M}^{(j)}}(\gamma, (\tau, z)) := (c\tau + d)^{-k} e^{2\pi i Tr(\mathcal{M}^{(j)}(-z^t \frac{c}{c\tau+d}z + \lambda^t \tau \lambda + 2\lambda^t z + \lambda^t \mu))}.$$

Let us define the usual slash operator on a function $f : \mathcal{H} \times \mathbb{C}^j \to \mathbb{C}$:

$$(f|_{\omega,k,\mathcal{M}^{(j)}}\gamma)(\tau, z) := \omega(\gamma) j_{k,\mathcal{M}^{(j)}}(\gamma, (\tau, z)) f(\gamma(\tau, z)), \ \gamma \in \Gamma^J,$$

where $\omega(\gamma)$ is the multiplier system of weight k on Γ^J so that it satisfies

$$\omega(\gamma_1\gamma_2)j_{k,\mathcal{M}^{(j)}}(\gamma_1\gamma_2,(\tau,z)) = \omega(\gamma_1)\omega(\gamma_2)j_{k,\mathcal{M}^{(j)}}(\gamma_1,\gamma_2(\tau,z))j_{k,\mathcal{M}^{(j)}}(\gamma_2,(\tau,z)),$$

for all $\gamma_1,\gamma_2 \in \Gamma^J$. Then one checks the following consistency condition (see also [10], Section I.1):

$$(f|_{\omega,k,\mathcal{M}^{(j)}}\gamma|_{\omega,k,\mathcal{M}^{(j)}}\gamma')(\tau,z) = (f|_{\omega,k,\mathcal{M}^{(j)}}\gamma\gamma')(\tau,z), \; \gamma,\gamma' \in \Gamma^J.$$

Throughout this paper we let

$$f|_{\omega,k,\mathcal{M}^{(j)}}\gamma = f|_\omega\gamma$$

unless it is specified. Also when ω is trivial, i.e. $\omega(\gamma) = 1$, for all $\gamma \in \Gamma^J$ we denote it as

$$f|_{\omega,k,\mathcal{M}^{(j)}}\gamma = f|_{k,\mathcal{M}^{(j)}}\gamma = f|\gamma.$$

Throughout this paper we let $v := Im(\tau)$, $y := Im(z) = (Im(z_1),\cdots,Im(z_j))$.

3. Jacobi Integral

Let $\mathcal{M}^{(j)}$ be fixed and $\mathcal{P}_{\mathcal{M}^{(j)}}$ be the space of functions f holomorphic in $\mathcal{H} \times \mathbb{C}^j$ which satisfy the growth condition

$$|f(\tau,z)| < K(|\tau|^\rho + v^{-\sigma})e^{2\pi Tr(\frac{\mathcal{M}^{(j)}y^ty}{v})},$$

for some positive constants K, ρ and σ.

Proposition 3.1. *The set $\mathcal{P}_{\mathcal{M}^{(j)}}$ has the following properties:*

(1) It is preserved under $|_{\omega,k,\mathcal{M}^{(j)}}$ for any real k and any $\gamma \in \Gamma^J$.
(2) It forms a vector space over $\mathbb{C}$.

Proof For simplicity we may assume that $j = 1$ and $\mathcal{M}^{(j)} = m \in \frac{1}{2}\mathbb{Z}$.
(1) It is enough to check for $[S,(0,0)], [T,(0,0)]$ and $[I,(\lambda,\mu)]$, where $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ since $\Gamma(1)^J$ is generated by those elements (see Section 4):

(a) For $[S, (0,0)]$,

$$|(f|_{\omega,k,m}[S, (0,0)])(\tau, z)| = |f(\tau + 1, z)|$$
$$< \quad K(|\tau + 1|^\rho + v^{-\sigma})e^{2\pi m\frac{y^2}{v}}$$
$$< \quad K'(|\tau|^{\rho'} + v^{-\sigma'})e^{2\pi m\frac{y^2}{v}}$$

for some positive constants K', ρ' and σ'.

(b) For $[T, (0,0)]$,

$$|(f|_{\omega,k,m}[T, (0,0)])(\tau, z)| = |(\tau)^{-k}e^{2\pi mi(\frac{-z^2}{\tau})}f(-\frac{1}{\tau}, \frac{z}{\tau})|$$
$$< \quad |\tau|^{-k}K(|\tau|^{-\rho} + (\frac{v}{|\tau|^2})^{-\sigma})|e^{2\pi mi(\frac{-z^2}{\tau})}|e^{2\pi m\frac{Im(z\bar{\tau})^2}{v|\tau|^2}}$$
$$< \quad K'(|\tau|^{\rho'} + v^{-\sigma'})e^{2\pi m\frac{y^2}{v}}$$

for some positive constants K', ρ' and σ'.

(c) For $[I, (\lambda, \mu)]$,

$$|(f|_{\omega,k,m}[I, (\lambda, \mu)])(\tau, z)| = |e^{2\pi mi(\lambda^2\tau + 2\lambda z)}f(\tau, z + \lambda\tau + \mu)|$$
$$< \quad K(|\tau|^\rho + v^{-\sigma})|e^{2\pi mi(\lambda^2\tau + 2\lambda z)}|e^{2\pi m\frac{(y + \lambda v)^2}{v}}$$
$$< \quad K(|\tau|^\rho + v^{-\sigma})e^{2\pi m\frac{y^2}{v}}.$$

(2) For any $f, g \in \mathcal{P}_m$ note that $f(\tau, z)e^{-2\pi m\frac{y^2}{v}}$ and $g(\tau, z)e^{-2\pi m\frac{y^2}{v}}$ satisfy the following growth condition

$$(3.1) \qquad\qquad |f(\tau, z)e^{-2\pi m\frac{y^2}{v}}| < K_1(|\tau|^{\rho_1} + v^{-\sigma_1}),$$

$$|g(\tau, z)e^{-2\pi m\frac{y^2}{v}}| < K_2(|\tau|^{\rho_2} + v^{-\sigma_2}),$$

for some positive constants K_i, ρ_i and $\sigma_i, i = 1, 2$. We conclude that $f + g \in \mathcal{P}_m$. $\qquad\square$

More generally, let $\mathcal{P}$ be the space of functions which are holomorphic in $\mathcal{H}$ with the growth condition in (3.1). It is easy to verify that $\mathcal{P}$ is preserved under $|_{\omega,k,0}$ for any $\gamma \in \Gamma^J$ and forms a ring.

Definition 3.2. (Jacobi Integral)

(1) A real analytic periodic function $f : \mathcal{H} \times \mathbb{C}^j \to \mathbb{C}$ is called a **Jacobi Integral** of weight $k \in \frac{1}{2}\mathbb{Z}$ and index $\mathcal{M}^{(j)}$ with multiplier system ω and a holomorphic period functions P_γ on Γ^J if it satisfies the following relations:

(i) For all $\gamma \in \Gamma^J$

(3.2)
$$(f|_{\omega,k,\mathcal{M}^{(j)}}\gamma)(\tau, z) = f(\tau, z) + P_\gamma(\tau, z),$$

where P_γ is in $\mathcal{P}_{\mathcal{M}^{(j)}}$.

(ii) It satisfies a growth condition, when $v, y \to \infty$,

$$|f(\tau, z)| v^{-\frac{k}{2}} e^{2\pi Tr(\mathcal{M}^{(j)}\frac{y^t y}{v})} \to 0.$$

(2) The space of Jacobi integrals forms a vector space over $\mathbb{C}$ and we denote it as $J^{\int}_{\omega,k,\mathcal{M}^{(j)}}(\Gamma^J)$. In particular when $j = 1$ we let $\mathcal{M}^{(j)} = m$ so the space is denoted by $J^{\int}_{\omega,k,m}(\Gamma^J)$.

Remark 3.3. (1) The periodic condition on f is equivalent to say that

$$f|_{\omega,k,\mathcal{M}^{(j)}}[I, (0, 1_n)] = f, f|_{\omega,k,\mathcal{M}^{(j)}}[S^\ell, (0, 0)] = f,$$

or

$$P_{[I,0,1_n]}(\tau, z) = 0, P_{[S^\ell,0,0]}(\tau, z) = 0,$$

for some $\ell \in \mathbb{Z}$ and for all $n = 1, 2, \cdots, j$ where $1_n \in \mathbb{Z}^j$ whose nth component is 1 and all other components are 0.

(2) The collection of holomorphic functions $\{P_\gamma | \gamma \in \Gamma^J\}$ occurring in (3.2) is called the system of **period functions** of f. The period functions P_γ satisfy the following consistency condition:

$$P_{\gamma_1\gamma_2} = P_{\gamma_1}|_{\omega,k,\mathcal{M}^{(j)}}\gamma_2 + P_{\gamma_2}, \text{ for all } \gamma_1, \gamma_2 \in \Gamma^J.$$

(3) If $P_\gamma(\tau, z) = 0$, for all $\gamma \in \Gamma^J$, then f is a usual **Jacobi form**, whose space will be denoted by $J_{\omega,k,\mathcal{M}^{(j)}}(\Gamma^J)$.

3.1. **Lifting from Jacobi integrals to Jacobi forms.** In this section we take $j = 1$ and $\mathcal{M}^{(j)} = m$ for simplicity. The result in this section can be extended to general $j \geq 1$ without any technical difficulties.

Let us define the following operator Ψ on $J^{\int}_{\omega,k,m}(\Gamma^J)$ as

$$\Psi(f)(\tau, z) := v^k e^{-4\pi m \frac{y^2}{v}} \left(\frac{\partial f}{\partial \bar{z}}\right)(\tau, z).$$

Then the following holds:

Proposition 3.4. Let $G(-\bar{\tau}, \bar{z}) := \Psi(f(\tau, z))$ with $f(\tau, z) \in J^{\int}_{k,m}(\Gamma^J)$. Then $G(\tau, z)$ is in $J_{1-k,-m}(\Gamma^J)$.

Proof (1) Let $\gamma = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (0, 0)] \in \Gamma^J$. By the definition of Jacobi integral, we have

$$(c\tau + d)^{-k} e^{2\pi i m \frac{-cz^2}{c\tau+d}} f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = f(\tau, z) + P_\gamma(\tau, z).$$

Then since $f(\tau, z)$ is real analytic with respect to z, we see that

$$(c\tau + d)^{-k} e^{2\pi i m \frac{-cz^2}{c\tau+d}} \frac{\partial f}{\partial \bar{z}}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \frac{1}{c\bar{\tau} + d} = \frac{\partial f}{\partial \bar{z}}(\tau, z).$$

From this it follows that

$$\Psi(f)\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$
$$= \frac{v^k}{|c\tau + d|^{2k}} e^{-4\pi m \frac{(Im(z(c\bar{\tau}+d)))^2}{v|c\tau+d|^2}} \frac{\partial f}{\partial \bar{z}}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$
$$= \frac{v^k}{|c\tau + d|^{2k}} e^{-4\pi m \frac{(Im(z(c\bar{\tau}+d)))^2}{v|c\tau+d|^2}} (c\tau + d)^k (c\bar{\tau} + d) e^{2\pi i m \frac{cz^2}{c\tau+d}} \frac{\partial f}{\partial \bar{z}}(\tau, z)$$
$$= \frac{v^k}{(c\bar{\tau} + d)^{k-1}} e^{2\pi i m \frac{c\bar{z}^2}{c\bar{\tau}+d}} \frac{\partial f}{\partial \bar{z}}(\tau, z)$$
$$= (c\bar{\tau} + d)^{1-k} e^{2\pi i m \frac{c\bar{z}^2}{c\bar{\tau}+d}} \Psi(f)(\tau, z).$$

Since $G(-\bar{\tau}, \bar{z}) = \Psi(f)(\tau, z)$, we have

$$
\begin{aligned}
& G(\frac{a\tau + d}{c\tau + d}, \frac{z}{c\tau + d}) \\
= \ & \Psi(f)(-\frac{a\bar{\tau} + b}{c\bar{\tau} + d}, \frac{\bar{z}}{c\bar{\tau} + d}) \\
= \ & \Psi(f)(\frac{a(-\bar{\tau}) - b}{-c(-\bar{\tau}) + d}, \frac{\bar{z}}{-c(-\bar{\tau}) + d}) \\
= \ & (-c(-\tau) + d)^{1-k} e^{2\pi im \frac{-cz^2}{-c(-\tau)+d}} \Psi(f)(-\bar{\tau}, \bar{z}) \\
= \ & (c\tau + d)^{1-k} e^{2\pi i(-m) \frac{cz^2}{c\tau + d}} G(\tau, z).
\end{aligned}
$$

(2) Let $\gamma = [I, (\lambda, \mu)] \in \Gamma^J$. By the definition of Jacobi integral, we have

$$
e^{2\pi im(\lambda^2 \tau + 2\lambda z)} \frac{\partial f}{\partial \bar{z}}(\tau, z + \lambda\tau + \mu) = \frac{\partial f}{\partial \bar{z}}(\tau, z).
$$

From this it follows that

$$
\begin{aligned}
& \Psi(f)(\tau, z + \lambda\tau + \mu) \\
= \ & v^k e^{-4\pi m \frac{(y + \lambda v)^2}{v}} \frac{\partial f}{\partial \bar{z}}(\tau, z + \lambda\tau + \mu) \\
= \ & v^k e^{-4\pi m \frac{(y + \lambda v)^2}{v}} e^{-2\pi im(\lambda^2 \tau + 2\lambda z)} \frac{\partial f}{\partial \bar{z}}(\tau, z) \\
= \ & e^{-2\pi im(\lambda^2 \bar{\tau} + 2\lambda \bar{z})} \Psi(f)(\tau, z).
\end{aligned}
$$

Since $G(\tau, z) = \Psi(f)(-\bar{\tau}, \bar{z})$, we have

$$
\begin{aligned}
& G(\tau, z + \lambda\tau + \mu) \\
= \ & \Psi(f)(-\bar{\tau}, \bar{z} + \lambda\bar{\tau} + \mu) \\
= \ & \Psi(f)(-\bar{\tau}, \bar{z} - \lambda(-\bar{\tau}) + \mu) \\
= \ & e^{-2\pi im(\lambda^2(-\tau) - 2\lambda z)} \Psi(f)(-\bar{\tau}, \bar{z}) \\
= \ & e^{-2\pi i(-m)(\lambda^2 \tau + 2\lambda z)} G(\tau, z).
\end{aligned}
$$

So the proof is completed. $\square$

3.2. **Examples.** We give several examples of real analytic Jacobi forms, whose holomorphic part or non holomorphic part can be regarded as Jacobi integrals.

The first example is from that in Zwegers[28]:

Example 3.5. For $z = x + iy \in \mathbb{C}$ and $\tau = u + iv \in \mathcal{H}$, consider the series

$$R(\tau, z) = \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} \{sgn(\nu) - E((\nu + \frac{y}{v})\sqrt{2v})\}(-1)^{\nu - \frac{1}{2}} e^{-\pi i \nu^2 \tau - 2\pi i \nu z},$$

$$E(z) = sgn(z)(1 - \beta(z^2)), \ \beta(x) = \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u} \, du \ (x \in \mathbb{R} \geq 0).$$

Take a multiplier system

$$\omega([I, (1,0)]) = \omega([I, (0,1)]) = -1,$$

$$\omega([S, (0,0)]) = e^{\frac{\pi i}{4}}, \omega([T, (0,0)]) = \frac{-1}{\sqrt{-i}}.$$

Then

(1) $(R|_{\omega, \frac{1}{2}, -\frac{1}{2}}[T, (0,0)])(\tau, z) = R(\tau, z) + P_{[T,(0,0)]}(\tau, z)$, where

$$P_{[T,(0,0)]}(\tau, z) = \int_{\mathbb{R}} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{\cosh \pi x} \, dx.$$

(2) For $a \in (0,1), b \in \mathbb{R}$ and $\tau \in \mathcal{H}$, let

$$R_{a,b}(\tau) = -i \int_{-\overline{\tau}}^{i\infty} \frac{g_{a,-b}(\tau)}{\sqrt{-i(z + \tau)}} dz,$$

where

$$g_{a,b}(\tau) := \sum_{\nu \in a + \mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b}.$$

Then

$$R_{a,b}(\tau) = i e^{-\pi i (a - \frac{1}{2})^2 \tau - 2\pi i (a - \frac{1}{2})b} R(\tau, (a - \frac{1}{2})\tau + b + \frac{1}{2}).$$

(3) This is a real analytic Jacobi integral of weight $\frac{1}{2}$ and index $-\frac{1}{2}$ with multiplier system ω.

(4) Furthermore, $\Psi(R)(\tau, z) \ (= v^{\frac{1}{2}} e^{2\pi \frac{y^2}{v}} (\frac{\partial R}{\partial \overline{z}})(\tau, z)) = \sqrt{2}\theta(-\overline{\tau}, \overline{z})$, where $\theta(\tau, z)$ is the well-known Jacobi Theta series defined as

$$\theta(\tau, z) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu (z + \frac{1}{2})}.$$

Example 3.6. The following is a real analytic Jacobi Eisenstein series[6]:

$$-\frac{1}{12}E_{2,1}^*(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\4n\geq r^2}} H(4n-r^2)q^n\xi^r + \frac{2}{\sqrt{v}}\sum_{r,f\in\mathbb{Z}}\beta(\pi f^2 v)q^{\frac{r^2-f^2}{4}}\xi^r$$

is a real analytic Jacobi form (Eisenstein series of weight 2 and index 1 on $\Gamma(1)^J$). Here, $H(n)$ denotes the Hurwitz class number formula (see [6]).

(1) Then $H^*(\tau,z) := \sum_{\substack{n,r\in\mathbb{Z}\\4n\geq r^2}} H(4n-r^2)q^n\xi^r$ is a (holomorphic) Jacobi integral of weight 2 and index 1 with period function

$$(3.3)\qquad P_{[T,(0,0)]}(\tau,z) = \frac{\sqrt{2}}{8\pi i}\int_0^{i\infty}\theta(t,0)(-i(t+\tau))^{-\frac{3}{2}}dt\cdot\theta(\tau,z).$$

(2) $R(\tau,z) := \frac{2}{\sqrt{v}}\sum_{r,f\in\mathbb{Z}}\beta(\pi f^2 v)q^{\frac{r^2-f^2}{4}}\xi^r$ is a (real analytic) Jacobi integral of weight 2 and index 1 with the same period function $P_{[T,(0,0)]}(\tau,z)$ in (3.3).

(3) It was shown[12] that $R(\tau,z) = \frac{\sqrt{2}}{8\pi i}\int_{-\tau}^{i\infty}\theta(t,0)(-i(t+\tau))^{-\frac{3}{2}}dt\cdot\theta(\tau,z)$.

The Higher-Level Appell functions studied in [25] can be regarded as Jacobi integral. For example it has the following property:

Example 3.7. Let

$$\mathcal{K}_1(q,x,y) = \sum_{m\in\mathbb{Z}}\frac{q^{\frac{m^2}{2}}x^m}{1-xyq^m}, q = e^{2\pi i\tau}, x = e^{2\pi iz}, y = e^{2\pi iw}, \tau\in\mathcal{H}, z,w\in\mathbb{C}.$$

Further $G(\tau,z,w) := \frac{1}{\theta(\tau,z)}\cdot\mathcal{K}_1(\tau,z,w)$ with $\theta(\tau,z) = \sum_{\lambda\in\mathbb{Z}}q^{\frac{\lambda^2}{2}}x^\lambda$. Note that

$$\frac{e^{-\pi i\frac{z^2}{\tau}}}{\sqrt{-i\tau}}\theta(-\frac{1}{\tau},\frac{z}{\tau}) = \theta(\tau,z).$$

Take a multiplier system

$$\omega([I,(1,0)]) = \omega([I,(0,1)]) = 1,$$

$$\omega([T,(0,0)]) = \sqrt{-i}$$

and let $\mathcal{M}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. Then it was shown that

$$(G|_{\omega,\frac{1}{2},\mathcal{M}^{(2)}}[T,(0,0)])(\tau,z,w) = G(\tau,z,w) + P_{[T,(0,0)]}(\tau,z,w),$$

where

$$P_{[T,(0,0)]}(\tau, z, w) = -\frac{1}{2}e^{\pi i \frac{w^2}{\tau}} \int_{\mathbb{R}} e^{-\pi x^2} \frac{e^{-2\pi i x \frac{w}{\sqrt{-i\tau}}} dx}{1 - e^{-2\pi x \sqrt{-i\tau}}}.$$

So $G(\tau, z, w)$ is a Jacobi integral with weight $\frac{1}{2}$ and index $\frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ with a period function $P_{[T,(0,0)]}(\tau, z, w)$.

4. Period Relations

In this section we study more precise period relations of Jacobi integral in $J_{\omega,k,m}(\Gamma(1)^J)$.

4.1. Jacobi Group. Let us introduce the following notations:

$$G_0 = [S, (0,0)], G_1 = [S, (1,0)], G_2 = [T, (1,0)],$$

$$G_3 = [I, (1,0)], G_4 = [I, (0,1)], I^J = [I, (0,0)],$$

$$V = G_2^3 G_1 = [-TS, (1,-1)], R = G_2^3 G_0 = [-TS, (0,-1)].$$

We recall the following facts:

Theorem 4.1. (1) $\Gamma(1)^J$ is generated by G_1 and G_2.
  (2) $\Gamma(1)^J$ is generated by G_0 and G_2.
  (3) $\Gamma(1)^J$ is generated by G_2 and V. The generators G_2 and V satisfy the relations

$$G_2^4 = V^3 = I^J,$$

$$VG_2^2 = [I, (-1,-2)]G_2^2 V = G_2^2[I, (1,2)]V = G_2^2 V[I, (-2,-1)],$$

  and these are the defining relations for $\Gamma(1)^J$.
  (4) $\Gamma(1)^J$ is generated by G_2 and R.

Proof See [4]. □

Corollary 4.2. The generators G_1 and G_2 of the group $\Gamma(1)^J$ satisfy the relations

$$G_2^4 = (G_2^3 G_1)^3 = I^J, \ G_2^4 = (G_2^3 G_0)^3 = I^J, \ G_2^4 = R^3 = I^J.$$

Proof See [4]. □

Remark 4.3. The following relations hold:

$$[I, (0, -1)] = G_0^{-1} G_2^{-2} G_0 G_2^2, \ [I, (0, -1)] = G_0^{-1} G_2^{-2} G_0,$$

$$[T, (0, 0)] = [-I, (1, 0)] G_2^3, \ [-I, (1, 0)] = G_0^{-1} G_2^{-2} G_0.$$

Definition 4.4. (parabolic element) We call any element of the form $[\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix} \right),$ $(0, r)]$, $\ell, r \in \mathbb{Z}$, a *"parabolic element"*.

4.2. **Period functions.** Classically, there are two relations period polynomial $p(\tau)$ associated with elliptic cusp forms of weight $2 - k$ should satisfy (see [13]), namely,

$$p(\tau) + \tau^{-k} p(-\frac{1}{\tau}) = 0$$

and

$$p(\tau) + \tau^{-k} p(\frac{\tau - 1}{\tau}) + (\tau - 1)^{-k} p(\frac{-1}{\tau - 1}) = 0.$$

In this section we study the relations in which the period function $P_{[T, 0, 0]}(\tau, z)$ associated with Jacobi integrals should satisfy. In particular when $z = 0$ we recover those period relations from elliptic modular forms.

For simplicity we consider the case when $j = 1$ and $\mathcal{M}^{(j)} = m$.

Proposition 4.5. *The transformation formulas of Jacobi integral on $\Gamma(1)^J$ in (3.3) can be reduced to the following two relations:*

  (1) $f|_{\omega, k, m}[S, (0, 0)] = f$,
  (2) $f|_{\omega, k, m}[T, (1, 0)] = f + P_{[T, (1, 0)]}$, with $P_{[T, (1, 0)]} \in \mathcal{P}_m$.

Proof Since $\Gamma(1)^J$ is generated by G_0 and G_2 the result follows. $\square$

Theorem 4.6. (Period Relations) *Take a multiplier system ω with $\omega(-I) = 1$. If a Jacobi integral f is even and periodic with respect to z, i.e., $P_{[-I, (0, 0)]}(\tau, z) = P_{[I, (0, 1)]}(\tau, z) = 0$, then the period functions $P_{[T, (0, 0)]}(\tau, z)$ and $h(\tau, z) := P_{[I, (1, 0)]}(\tau, z)$ satisfy the following properties:*

  (1) $P_{[T, (1, 0)]} = P_{[T, (0, 0)]}$.
  (2) $P_{[T, (0, 0)]} + P_{[T, (0, 0)]}|_{\omega, k, m}[T, (0, 0)] = 0$.
  (3) $P_{[T, (0, 0)]} + P_{[T, (0, 0)]}|_{\omega, k, m}[ST, (0, 0)] + P|_{\omega, k, m}[(ST)^2, (0, 0)] = 0$.

(4) $h|_{\omega,k,m}[T,(0,0)] = -P_{[T,(0,0)]} + P_{[T,(0,0)]}|_{\omega,k,m}[I,(0,-1)]$.

(5) $h = P_{[T,(0,0)]} - P_{[T,(0,0)]}|_{\omega,k,m}[-I,(1,0)]$.

(6) $h + h|_{\omega,k,m}[-I,(1,0)] = 0$.

Proof Note that $P_{[\gamma,(\lambda,\mu)]} = P_{[-\gamma,(\lambda,\mu)]}$ since f is even with respect to z.

(1) $f|_{\omega,k,m}[I,(0,-1)]|_{\omega,k,m}[T,(0,0)] = f + P_{[T,(0,0)]} = f + P_{[T,(1,0)]}$.

(2) It follows from $[T,(0,0)]^2 = [-I,(0,0)]$.

(3) It follows from $[ST,(0,0)]^3 = [-I,(0,0)]$.

(4) From $[I,(1,0)][T,(0,0)] = [T,(0,-1)] = [T,(0,0)][I,(0,-1)]$ it follows that

$$h|_{\omega,k,m}[T,(0,0)] + P_{[T,(0,0)]} = P_{[I,(0,-1)]}.$$

(5) It follows from $[T,(1,0)] = [T,(0,0)][I,(1,0)]$ and (1).

(6) By the definition of h

$$\begin{aligned} h|_{\omega,k,m}[-I,(1,0)] &= P_{[I,(1,0)]}|_{\omega,k,m}[-I,(1,0)] \\ &= P_{[-I,(0,0)]} - P_{[-I,(1,0)]} = -h. \quad \square \end{aligned}$$

Example 4.7. The following examples are in Example 3.5: We set

$$P(\tau,z) := P_{[T,(0,0)]}(\tau,z) = \int_{\mathbb{R}} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{\cosh \pi x} \, dx$$

and

$$h(\tau,z) := P_{[I,(1,0)]}(\tau,z) = 2e^{-\pi i z - \pi i \tau/4}.$$

Then we have

(1) $P(\tau,z) - (P|_{\omega}[I,(0,1)])(\tau,z)$
$= P(\tau,z) + P(\tau,z+1) = \frac{2}{\sqrt{-i\tau}} e^{\pi(z+\frac{1}{2})^2/\tau} = (h|_{\omega}[T,(0,0)])(\tau,z+1)$.

(2) $P(\tau,z) - (P|_{\omega}[I,(1,0)])(\tau,z)$
$= P(\tau,z) + e^{-2\pi i z - \pi i \tau} P(\tau,z+\tau) = 2e^{-\frac{\pi i \tau}{4} - \pi i z} = h(\tau,z)$.

(3) $P(\tau,z) + (P|_{\omega}[T,(0,0)])(\tau,z)$
$= P(\tau,z) - \frac{1}{\sqrt{-i\tau}} e^{2\pi i \frac{1}{2}\frac{z^2}{\tau}} P(\tau,z) = 0$.

(4) $(P|_{\omega}[T,(0,0)])(\tau,z) + (P|_{\omega}[S,(0,0)])(\tau,z) + (P|_{\omega}[STS,(0,0)])(\tau,z)$
$= -P(\tau,z) + e^{\frac{\pi i}{4}} P(\tau+1,z) + e^{-\frac{\pi i}{4}} \frac{e^{\pi i z^2/(\tau+1)}}{\sqrt{\tau+1}} P(\frac{\tau}{\tau+1}, \frac{z}{\tau+1}) = 0$.

Example 4.8. The following examples are in Example 3.7: We set

$$P(\tau, z, w) := P_{[T,((0,0),(0,0))]}(\tau, z, w) = e^{\pi i \frac{w^2}{\tau}} \Phi(\tau, w)$$

and

$$h(\tau, z, w) := P_{[I,((0,0),(1,0))]}(\tau, z, w) = e^{-2\pi i w - \pi i \tau}.$$

Here,

$$\Phi(\tau, w) := -\frac{1}{2} \int_{\mathbb{R}} e^{-\pi x^2} \frac{e^{-2\pi i x \frac{w}{\sqrt{-i\tau}}} dx}{1 - e^{-2\pi x \sqrt{-i\tau}}}.$$

Then we have

(1) $P(\tau, z, w) - (P|_\omega[I, ((0,0),(0,-1))])(\tau, z, w)$
$= e^{\pi \frac{w^2}{\tau}} \Phi(\tau, z, w) - e^{\pi i \frac{(w-1)^2}{\tau}} \Phi(\tau, w - 1)$
$= -\frac{i}{\sqrt{-i\tau}} e^{\pi i \frac{(w-1)^2}{\tau}} = -h|_w[T, ((0,0),(0,0))].$

(2) $P(\tau, z, w) - (P|_\omega[I, ((0,0),(1,0))])(\tau, z, w)$
$= e^{\pi i \frac{w^2}{\tau}} \Phi(\tau, z, w) - e^{\pi i \frac{w^2}{\tau}} \Phi(\tau, z, w + \tau) = e^{-2\pi i w - \pi i \tau} = h(\tau, z, w).$

(3) $P(\tau, z, w) + (P|_\omega[T, ((0,0),(0,0))])(\tau, z, w)$
$= e^{\pi i \frac{w^2}{\tau}} \Phi(\tau, z, w) + \frac{\sqrt{-i}}{\sqrt{\tau}} \Phi(-\frac{1}{\tau}, \frac{z}{\tau}, \frac{w}{\tau}) = -1.$

Proposition 4.9. *Proposition 4.6-(2) implies that Proposition 4.6-(5)*

Proof From the relation $[ST, (1,0)][ST, (0,0)][ST, (0,0)] = [-I, (0,-1)]$ we derive

$$h|_{\omega,k,m}[-I, (0,0)] = -P|_{\omega,k,m}[ST, (0,0)] - P|_{\omega,k,m}[(ST)^2, (0,0)]$$
$$-P|_{\omega,k,m}[I, (1,0)][-I, (0,0)] = 0. \quad \square$$

Using the above relation we study a family of Jacobi integral which has a theta decomposition

4.3. **A Jacobi integral with Theta decomposition.** Consider a holomorphic Jacobi integral $\phi_{k,m} : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ such that

(A) $(\phi_{k,m}|_{\omega,k,\mathcal{M}}[S, ((0,0),(0,0))])(\tau, z, w) = \phi_{k,m}(\tau, z, w).$
(B) $(\phi_{k,m}|_{\omega,k,\mathcal{M}}[I, ((1,0),(1,0))])(\tau, z, w) = \phi_{k,m}(\tau, z, w).$
(C) $(\phi_{k,m}|_{\omega,k,\mathcal{M}}[I, ((0,0),(0,1))])(\tau, z, w) = \phi_{k,m}(\tau, z, w).$

Then the following holds:

Proposition 4.10. (1) $\phi_{k,m}(\tau, z, w) = \sum\limits_{\ell \ (mod\ 2m)} h_\ell(\tau, w)\theta_{m,\ell}(\tau, z),$

$\quad$ where $h_\ell(\tau, w) = e^{-\frac{\pi i \ell^2 \tau}{2m}} \int_p^{p+1} \phi(\tau, z, w)e^{-2\pi i \ell z}dz, p \in \mathbb{C}$ and

$$\theta_{m,\ell}(\tau, z) := \sum\limits_{\substack{r \in \mathbb{Z} \\ r \equiv \ell (mod 2m)}} q^{\frac{r^2}{4m}}\xi^r, q = e^{2\pi i \tau}, \xi = e^{2\pi i z}.$$

(2) $(\phi_{k,m}|_{\omega,k,\mathcal{M}}[T, ((0,0),(0,0))])(\tau, z, w) = \phi_{k,m}(\tau, z, w) + \sum\limits_{\ell=0}^{2m-1} P_\ell(\tau, w)\theta_{m,\ell}(\tau, z),$

$\quad$ where $P_\ell(\tau, w) = h_\ell(\tau, w) - (h_\ell|_{\omega k - \frac{1}{2},\mathcal{M}}[T, (0,0)])(\tau, w).$

(3) $(\phi_{k,m}|_{\omega,k,\mathcal{M}}[I, ((0,1),(0,0))])(\tau, z, w) = \phi_{k,m}(\tau, z, w) + \sum\limits_{\ell=0}^{2m-1}(P_\ell(\tau, w)$

$\quad -(P_\ell|_{\omega,k-\frac{1}{2},\mathcal{M}}[-I, ((1,0),(0,0))])(\tau, w))\theta_{m,\ell}(\tau, z).$

Proof (1) The condition (B) implies that

$$e^{2\pi i Tr(\mathcal{M}(\binom{1}{0})\tau(1,0)+2\binom{1}{0})(z,w))}\phi_{k,m}(\tau, z + \tau, w) = \phi_{k,m}(\tau, z, w)$$

so that it has the following theta series expansion,

$$\phi_{k,m}(\tau, z, w) = \sum\limits_{\ell \ (mod\ 2m)} h_\ell(\tau, w)\theta_{m,\ell}(\tau, z)$$

(see detailed proof in [10] or [28]).

(2) This follows from transformation formula of Theta Series (see [10])

$$\theta_{m,\mu}(-\frac{1}{\tau}, \frac{z}{\tau}) = \sqrt{\frac{\tau}{2mi}}e^{2\pi i m \frac{z^2}{\tau}}\sum\limits_{\nu \ (mod\ 2m)} e^{-2\pi i \frac{\mu\nu}{2m}}\theta_{m,\nu}(\tau, z).$$

(3) This follows from the period relation given in Proposition 4.6: if

$$(\phi_{k,m}|_{\omega,k,\mathcal{M}}[I, ((0,1),(0,0))])(\tau, z, w) = \phi_{k,m}(\tau, z, w) + P_{[I,((0,1),(0,0))]}(\tau, z, w)$$

then

$$P_{[I,((0,1),(0,0))]}(\tau, z, w) = P_{[T,((0,0),(0,0))]}(\tau, z, w)$$
$$-(P_{[T,((0,0),(0,0))]}|_{\omega,k-\frac{1}{2},\mathcal{M}}[-I, ((1,0),(0,0))])(\tau, z, w).$$

$\square$

4.4. **Cohomology.** We call any collection of functions $\{\varphi_\gamma | \ \gamma \in \Gamma^J\}$ in $\mathcal{P}_m$ which satisfies

$$(4.1) \qquad \varphi_{\gamma_1 \gamma_2} = \varphi_{\gamma_1}|_{\omega,k,m}\gamma_2 + \varphi_{\gamma_2}, \text{ for } \gamma_1, \gamma_2 \in \Gamma^J$$

a *cocycle* of weight k and index m on Γ^J. A *coboundary* of weight k and index m on Γ^J is a *cocycle* $\{\varphi_\gamma | \ \gamma \in \Gamma^J\}$ such that $\varphi_\gamma = \varphi|_{\omega,k,m}\gamma - \varphi$ for all $\gamma \in \Gamma^J$, with φ a fixed function in $\mathcal{P}_m$. The parabolic cocycles on Γ^J are the cocycles $\{\varphi_\gamma | \ \gamma \in \Gamma^J\}$ which satisfy the following additional condition: for each parabolic element $Q_j, j = 0, ..., \ell$, there exist $\varphi_j, \in \mathcal{P}_m, j = 0, .., \ell$, such that

$$(4.2) \qquad \varphi_{Q_j} \ = \ \varphi_j|_{\omega,k,m}Q_j - \varphi_j.$$

Definition 4.11.   (1) The cohomology group $H^1_{\omega,k,m}(\Gamma, \mathcal{P}_m)$ is defined to be the vector space of cocycles modulo coboundaries.
   (2) Let $\tilde{H}^1_{\omega,k,m}(\Gamma, \mathcal{P}_m)$ be the subgroup of $H^1_{\omega,k,m}(\Gamma, \mathcal{P}_m)$ defined as *the space of parabolic cocycles* modulo coboundries and we call $\tilde{H}^1_{\omega,k,m}(\Gamma, \mathcal{P}_m)$ a parabolic cohomology group.

Remark 4.12.   (1) This is an analogous definition of the Eichler (parabolic) cohomoloy group $\tilde{H}^1_{k,v}(\Gamma, P_k)$, where P_k is the vector space of polynomials of degree $\leq k$ (see [14]).
   (2) For each $\phi \in J_{k,m}(\Gamma)$ there are at least two ways to attach the elements in $\tilde{H}^1_{\omega,k,m}(\Gamma, \mathcal{P}_m)$. One is via Eichler integral with $\mathbb{Q}$-division points and the other is via Eichler Integral and theta decomposition.

Now take $\Gamma^J = \Gamma(1)^J$ and consider the space of the following period functions:

$$(4.3)\mathcal{P}er_{\omega,k,m} \ := \ \{P \in \mathcal{P}_m \,|\, P + P|_{\omega,k,m}[T, (0,0)]$$
$$= P + P|_{\omega,k,m}[ST, (0,0)] + P|_{\omega,k,m}[(ST)^2, (0,0)] = 0\}.$$

Then the following is true:

Proposition 4.13. $\mathcal{P}er_{\omega,k,m}$ *is a generating set of all parabolic cocycles of* $\Gamma(1)^J$.

Proof First of all, one may regard $P \in \mathcal{P}er_{\omega,k,m}$ as the element $P = P_{[T,(0,0)]}$ in the set of parabolic cocycles on $\Gamma(1)^J$ from Proposition 4.6. On the other hand, the set $\{[S,(0,0)], [T,(0,0)], [I,(0,1)]\}$ generates $\Gamma(1)^J$ and, again from Proposition 4.6, one has

$$P_{[I,(0,1)]} = P_{[T,(0,0)]} - P_{[T,(0,0)]}|_{\omega,k,m}[-I,(1,0)].$$

So it is clear that $P_{[T,(0,0)]}$ generates every parabolic cocycle in $\Gamma(1)^J$. $\square$

5. **Jacobi Poincaré series and Existence of Jacobi Integral**

5.1. **Jacobi Poincaré series.** A generalized Poincaré series was studied in [14] to show the isomorphism between the parabolic cohomology group and space of elliptic modular cusp forms of the arbitrary weight.

Here we also introduce a generalized (Jacobi) Poincaré series to show the existence of Jacobi integral, which may has poles, associated to given period functions $P_\gamma \in \mathcal{P}_m$ on $\Gamma(1)^J$.

Definition 5.1. Suppose $\{\varphi_\gamma \,|\, \gamma \in \Gamma(1)^J\}$ is a parabolic cocycle of weight k and index m which satisfies the additional condition that $\varphi_{[S,(0,0)]} = \varphi_{[I,(0,1)]} = 0$. Suppose k is a positive even integer and ω is a multiplier system of weight k. Then the generalized Poincaré series $\Phi(\{\varphi_\gamma\}, k, m, \omega; \tau, z) = \Phi(\tau, z)$ is defined by

$$(5.1) \qquad \Phi(\tau, z) := \sum_{\gamma \in \Gamma(1)^J_\infty \backslash \Gamma(1)^J} (\varphi_\gamma|_{\omega,k,m}\gamma)(\tau, z),$$

where $\Gamma(1)^J_\infty = \{[\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), (0,\mu)]\,|\, n, \mu \in \mathbb{Z}\}$.

Note that the assumption $\varphi_{[S,(0,0)]} = \varphi_{[I,(0,1)]} = 0$ has been made to insure that the individual terms of the series are independent of the choice of γ of coset representatives.

Theorem 5.2. *For sufficiently large k ($> g$) and $m > 0$ the generalized Poincaré series $\Phi(\{\varphi_\gamma\}, k, m, \omega; \tau, z)$ converges absolutely where $g = max(2e + 5, 4)$, where e is defined in Lemma 8.2 in Appendix.*

Proof The proof of absolute convergence of the series defining $\Phi(\tau, z)$ is based upon a series of lemmas. Those are essentially Lemma 4, Lemma 5 and Lemma 6 in [14]. We give the detailed proof in the Appendix. $\square$

5.2. **Existence of Jacobi Integral.**

Theorem 5.3. *Let r be any real number, $m > 0$ and ω a multiplier system of weight r and index m. Suppose that $\{\varphi_\gamma | \ \gamma \in \Gamma(1)^J\}$ is a parabolic cocycle of weight k and index m in $\mathcal{P}_m$ such that $\varphi_{[S,(0,0)]} = \varphi_{[I,(0,1)]} = 0$. Then there is a meromorphic function $f : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that*

$$(f|_{\omega,r,m}\gamma)(\tau, z) - f(\tau, z) = \varphi_\gamma(\tau, z) \ \text{for all } \gamma \in \Gamma(1)^J.$$

Proof We take a generalized Poincaré series $\Phi(\{\varphi_\gamma\}, k, m, \omega; \tau, z) = \Phi(\tau, z)$ for sufficiently large k such that $k > g$, $g = max(2e + 5, 4)$. For $\gamma \in \Gamma(1)^J$, we see that

(5.2)
$$\Phi|_{\omega,r,m}\gamma = \bar{\omega}(\gamma)j_{k,m}(\gamma, (\tau, z))^{-1}\Phi(\tau, z) - \bar{\omega}(\gamma)j_{k,m}(\gamma, (\tau, z))^{-1}g(\tau, z)\varphi_\gamma(\tau, z),$$

where $g(\tau, z)$ is the Eisenstein series

(5.3)
$$g(\tau, z) := \sum_{\gamma \in \Gamma(1)_\infty \backslash \Gamma(1)^J} \omega(\gamma)j_{k,m}(\gamma, (\tau, z)).$$

The functional equation (5.2) is a straightforward consequence of the absolute convergence of (5.1), the consistency condition for the cocycle $\{\varphi_\gamma\}$ and the consistency condition for the multiplier system ω. For $k \geq 4$ the series in (5.3) converges absolutely and it follows that

$$g(\gamma(\tau, z)) = \bar{\omega}(\gamma)j_{k,m}(\gamma, (\tau, z))^{-1}g(\tau, z)$$

for $\gamma \in \Gamma^J$. Thus, putting $F(\tau, z) := -\frac{\Phi(\tau,z)}{g(\tau,z)}$ and applying (5.2), we find that

$$
\begin{aligned}
(F|_{\omega,r,m}\gamma)(\tau, z) &= -\frac{(\Phi|_{\omega,r,m}\gamma)(\tau, z)}{g(\gamma(\tau, z))} \\
&= -\frac{\Phi(\tau, z)}{g(\tau, z)} + \varphi_\gamma(\tau, z) = F(\tau, z) + \varphi_\gamma(\tau, z)
\end{aligned}
$$

so that F is a solution of the functional equation. $\square$

Remark 5.4. In [14] the generalized Poincaré series has been studied to show the isomorphism between the Eichler (parabolic) cohomology group $\tilde{H}^1_{k,v}(\Gamma, P_k)$ and the space of cusp forms of weight $2 - k$ on Γ, where the Petersson's result has been used to guarantee that one can construct a modular form which has the assigned poles and zeros in $\mathcal{H}$ (see [14] for details). However it is not known yet if the analogous result of Petersson can be extended to the Jacobi form case to show the constructed function f in Theorem 5.3 is holomorphic in $\mathcal{H} \times \mathbb{C}$.

6. Mock Jacobi forms

In this section we introduce a mock Jacobi form, which has a corresponding dual Jacobi form. Further study see [8].

6.1. Mock modular form.

The concept of Mock modular form, which was motivated from Ramanujan Mock Theta function, was first introduced by Zagier in [27]: A function $H : \mathcal{H} \to \mathbb{C}$ is called a **mock modular form** if

(1) It is holomorphic in $\mathcal{H}$ with only possible poles at the cusps (so that it contains the weakly holomorphic modular forms).

(2) There is a rational number λ such that $H(q), q = e^{2\pi i \tau}$, must be multiplied by q^λ in order to have any kind of modularity properties, and a "shadow" $g = \mathcal{S}[h]$ which is an ordinary modular form of weight $2 - k$ such that the holomorphic function $h(\tau) = q^\lambda H(q)$ becomes a non-holomorphic modular form of weight k when we complete it by adding a correction term $g^*(\tau)$ associated to $g(\tau)$.

(3) This "shadow" depends $\mathbb{R}$-linearly on h and vanishes if and only if h is a modular form, so that we have an exact sequence over $\mathbb{R}$:

$$0 \to \mathbf{M}^!_k \to \mathbb{M}_k \to^\mathcal{S} M_{2-k}$$

Here, $\mathbf{M}^!_k, \mathbb{M}_k$ and M_{2-k} are the space of weakly holomorphic modular forms, the space of mock modular forms and the space of modular forms, respectively.

Remark 6.1. A mock modular form defined here is more restricted than that in [27] since we take a rational invariant $\lambda = 0$ (see [27] for more detailed information).

The various examples were discussed by Zagier[27]. Here is one more example, which was already computed in [23].

Example 6.2.   (1) Assume $k \in 2\mathbb{Z}$ and let $\tau = u + iv \in \mathcal{H}$. Consider

$$G_k(\tau|s) = \frac{1}{2} \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} \Big(\frac{v}{|m\tau+n|^2}\Big)^s$$

which converges for $Re(s) > 1 - \frac{k}{2}$.

Then the Fourier expansions of $G_k(\tau|s)$ was derived in [23]:

$$G_k(\tau|s) = \zeta(k+2s)v^s + (-1)^{\frac{k}{2}}\pi\frac{2^{2-k-2s}\Gamma(k-1+2s)}{\Gamma(k+s)\Gamma(s)}\zeta(k-1+2s)v^{1-k-s}$$

$$+ \frac{(-1)^{\frac{k}{2}}(2\pi)^{k+2s}}{\Gamma(k+s)\Gamma(s)}v^s\sum_{n\geq 1}\sigma_{k-1+2s}(n)\{\sigma(4\pi nv, k+s, s)e^{2\pi in\tau}$$

$$+\sigma(4\pi nv, s, k+s)e^{-2\pi in\bar\tau}\}$$

where $\sigma_\omega(n) = \sum\limits_{\substack{d|n\\d>0}} d^\omega$ and $\sigma(\eta, \alpha, \beta) = \int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-\eta u}\,du$.

Let $G_k^*(\tau|s) = \pi^{-s}\Gamma(s)G_k(\tau|s)$. Then

$$G_k^*(\tau|0) = H_k^*(\tau) + R_k^*(\tau),$$

where

$$H_k^*(\tau) = \frac{(-k)!}{(2\pi i)^{-k}}\zeta(1-k) - \frac{(1-k)!}{(2\pi i)^{1-k}}\int_\tau^{i\infty}[G_{2-k}(w|0) - \zeta(2-k)](w-\tau)^{-k}dw$$

$$= \frac{(-k)!}{(2\pi i)^{-k}}\Big(\zeta(1-k) + \sum_{n\geq 1}\sigma_{k-1}(n)e^{2\pi in}\Big)$$

and

$$R_k^*(\tau) = (-k)!\zeta(2-k)\Big(\frac{v}{\pi}\Big)^{1-k}$$

$$- \frac{(1-k)!}{(2\pi i)^{1-k}}\int_{-\bar\tau}^{i\infty}[G_{2-k}(w|0) - \zeta(2-k)](w+z)^{-k}dw.$$

So $H_k^*(\tau)$ is a mock modular form with Shadow $\mathcal{S}[H_k^*(\tau)] = G_{2-k}(\tau|0)$.

6.2. **Mock Jacobi forms.** Let us recall the following heat operator introduced in [9]: Take a matrix $\mathcal{M}^{(j)} \in M_{j \times j}(\mathbb{R})$. The heat operator $L_{\mathcal{M}^{(j)}}$ is defined by

$$L_{\mathcal{M}^{(j)}} = 8\pi i |\mathcal{M}^{(j)}| \frac{\partial}{\partial \tau} - (\frac{\partial}{\partial z})^t \widetilde{\mathcal{M}^{(j)}} (\frac{\partial}{\partial z}),$$

where

$$(\frac{\partial}{\partial z}) = (\frac{\partial}{\partial z_\ell}), z = (z_\ell)_{1 \le \ell \le j} \in \mathbb{C}^j$$

and $|\mathcal{M}^{(j)}|$ is the determinant of $\mathcal{M}^{(j)}$, $\widetilde{\mathcal{M}^{(j)}} = (\widetilde{\mathcal{M}^{(j)}}_{mn})$, $\widetilde{\mathcal{M}^{(j)}}_{mn}$ is the cofactor of the (m,n)th entry of $\mathcal{M}^{(j)}$ for $j \ge 2$, and $\widetilde{\mathcal{M}^{(j)}} = 1$ when $j = 1$.

Definition 6.3. A mock Jacobi form $\phi : \mathcal{H} \times \mathbb{C}^j \to \mathbb{C}$ is a meromorphic Jacobi integral in $J^f_{\omega, -k+\frac{j}{2}, \mathcal{M}^{(j)}}(\Gamma^J), k \in \mathbb{Z}_{\ge 0}$, such that $L^{k+1}_{\mathcal{M}^{(j)}}(\phi)$ is a nontrivial (meromorphic) Jacobi form of weight $k + \frac{j}{2} + 2$ and index $\mathcal{M}^{(j)}$ with multiplier system. The Jacobi form $L^{k+1}_{\mathcal{M}^{(j)}}(\phi)$ is called a "dual" of ϕ. In other words, we say that a meromorphic Jacobi integral which has a "dual" Jacobi form is a mock Jacob form.

The following was introduced by Zwegers[28]:

6.3. **Lerch Sum.** Consider the Lerch sum,

$$\mu(\tau, z, w) := \frac{e^{\pi i w}}{\theta(\tau, z)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2 + n) \tau + 2\pi i n z}}{1 - e^{2\pi i n \tau + 2\pi i w}},$$

which was originally studied by Lerch and whose elliptic and modular transformation properties were derived by Zwegers[28] to connect with Mock theta function. Here

$$\theta(\tau, z) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu (z + \frac{1}{2})}.$$

Using the transformation properties of $\mu(\tau, z, w)$ and $\theta(\tau, z)$ the followings were derived in [28] and [2]:

Example 6.4.   (1) Let $f(\tau, z) := e^{\pi i z - \pi i \tau / 4} \mu(\tau, z, \frac{1}{2}\tau + \frac{1}{2})$ which is a Jacobi integral of a weight $\frac{1}{2}$ and an index $m = -\frac{1}{2}$. It satisfies the following transformation properties:

(a)

$$\frac{-i}{\sqrt{-i\tau}}e^{\pi\frac{z^2}{\tau}}f(-\frac{1}{\tau},\frac{z}{\tau}) = f(\tau,z) - \frac{1}{2i}e^{\pi i z - \pi i \tau/4}h(\tau,z-\frac{1}{2}\tau+\frac{1}{2}).$$

So, this implies that the period

$$P(\tau,z) := P_{[T,(0,0)]}(\tau,z) = -\frac{1}{2i}e^{\pi i z - \pi i \tau/4}h(\tau,z-\frac{1}{2}\tau+\frac{1}{2}).$$

(b) $L_m(f(\tau,z))$ is a "dual" of f, that is, a (nontrivial) (meromorphic) Jacobi form of weight $\frac{5}{2}$ and index $-\frac{1}{2}$. Here,

$$L_m := 4\pi i \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial z^2}$$

is the corresponding heat operator.

(2) More generally,let $f_{a,b}(\tau,z) := e^{2\pi i a z - \pi i a^2 \tau}\mu(\tau,z,a\tau+b)$, for any $a,b \in \mathbb{R}$, which is a Jacobi integral of weight $\frac{1}{2}$ and index $-\frac{1}{2}$ with its dual $L_m(f_{a,b})$ which is a Jacobi form of weight $\frac{5}{2}$ and index $-\frac{1}{2}$.

(3) In fact that the dual of $f_{a,b}$ was computed explicitly in [2]:

$$(4\pi i \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial z^2})(f_{a,b}(\tau,z)) = e^{2\pi i a z - \pi i a^2 \tau}\frac{16\pi^2\eta(\tau)^6}{\theta(\tau,a\tau+b)\theta(\tau,z)^3}$$

$$\times\{\alpha_1(\tau)\theta_0(2\tau,2z+a\tau+b) - \alpha_0(\tau)\theta_1(2\tau,2z+a\tau+b)\}.$$

Here,

$$\theta_0(\tau,z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

$$\theta_1(\tau,z) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

$$\alpha_0(\tau) = \alpha_0^{a,b} := \sum_{n\in\mathbb{Z}}(n+a/2)e^{2\pi i n^2 \tau + 2\pi i n(a\tau+b)},$$

$$\alpha_1(\tau) = \alpha_1^{a,b} := \sum_{n\in\frac{1}{2}+\mathbb{Z}}(n+a/2)e^{2\pi i n^2 \tau + 2\pi i n(a\tau+b)}.$$

Remark 6.5. Further a family examples via Eichler integrals are constructed in [8]

7. Conclusion

In this paper we study the period relations associated with Jacobi integral. This explains the relations from the Modell integral associated to Lerch sums[28] and from the functional relations associated higher Appell functions[25]. On the other hand, modular symbols can be studied purely algebraically using period relations[18] and recently modular symbols are extended to the complex weight forms associated to Maass wave forms. We are intending to develop higher modular symbols, Jacobi-modular symbols, using multi-variable period relations as well as the actions of Hecke operators on them[7].

8. Appendix

Here we begin to prove Theorem 5.2:

Lemma 8.1. *For real numbers c, d and $\tau = u + iv$, we have*

$$(\frac{v^2}{1+4|\tau|^2})(c^2 + d^2) \leq |c\tau + d|^2 \leq 2(|\tau|^2 + v^{-2})(c^2 + d^2).$$

If $A \in \Gamma(1)$ consider a factorization of A, $A = C_1 \cdots C_q$ where each C_i is T or a power of S. Eichler showed that for any $A \in \Gamma(1)$ the factorization can be carried out so that

$$q \leq m_1 log\mu(A) + m_2,$$

where $m_1, m_2 > 0$ are independent of A and

$$\mu(A) = a^2 + b^2 + c^2 + d^2 \text{ if } A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

We assume that the cocycle $\{\varphi_\gamma\}$ in $\mathcal{P}_m$ satisfies

$$(8.1) \qquad |\varphi_{[T,(0,0)]}(\tau, z)| \quad < \quad K(|\tau|^\rho + v^{-\sigma})e^{2\pi m \frac{y^2}{v}},$$

$$|\varphi_i(\tau, z)| \quad < \quad K(|\tau|^\rho + v^{-\sigma})e^{2\pi m \frac{y^2}{v}}, \text{ for } 0 \leq i \leq l.$$

Here $\varphi_i(\tau, z)$ is defined by (4.2) and K, ρ, σ are positive constants. Assume also $2\sigma > k, \rho > -k$.

Lemma 8.2. *If $\{\varphi_\gamma\}$ is a parabolic cocycle then there exists $K^* > 0$ such that*

$$|\varphi_{[C_h,(0,0)]}|_{\omega,k,m}[C_{h+1}, (0,0)] \cdots [C_q, (0,0)](\tau, z)| \leq K^*\mu(A)^e\{|\tau|^{6e+2k} + v^{-6e-2k}\}e^{2\pi m \frac{y^2}{v}},$$

for $1 \leq h \leq q$. *Here* $e = max(\frac{\rho}{2}, \sigma - \frac{k}{2})$ *and* $A = C_1 \cdots C_q$ *is a factorization of* $A \in \Gamma(1)$.

Proof Consider first the case when C_h is T. Let $\gamma = C_{h+1} \cdots C_q = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then, by (8.1),

$$|\varphi_{[C_h,(0,0)]|[\gamma,(0,0)]}(\tau, z)| = |c\tau + d|^{-k}|e^{2m\pi i(\frac{-cz^2}{c\tau+d})}||\varphi_{[C_h,(0,0)]}(\gamma\tau, \frac{z}{c\tau+d})|$$

$$< |c\tau + d|^{-k}K(|\gamma\tau|^\rho + v^{-\sigma}|c\tau + d|^{2\sigma})e^{2m\pi\frac{y^2}{v}}$$

$$= (K|a\tau + b|^\rho|c\tau + d|^{-k-\rho} + K|c\tau + d|^{2\sigma-k}v^{-\sigma})e^{2m\pi\frac{y^2}{v}}.$$

By Lemma 8.1,

$$|a\tau + b|^\rho \leq 2^{\frac{\rho}{2}}(|\tau|^2 + v^{-2})^{\frac{\rho}{2}}(a^2 + b^2)^{\frac{\rho}{2}},$$

$$|c\tau + d|^{2\sigma-k} \leq 2^{\sigma-\frac{k}{2}}(|\tau|^2 + v^{-2})^{\sigma-\frac{k}{2}}(a^2 + b^2)^{\sigma-\frac{k}{2}},$$

$$|c\tau + d|^{-k-\rho} \leq 2^{\sigma-\frac{k}{2}}(|\tau|^2 + v^{-2})^{\frac{-k-\rho}{2}}(a^2 + b^2)^{\frac{-k-\rho}{2}}.$$

Hence we have

$$|\varphi_{[C_h,(0,0)]}|[\gamma,(0,0)](\tau, z)| < e^{2\pi m\frac{y^2}{v}}\{K2^{\frac{\rho}{2}}(|\tau|^2 + v^{-2})^{\frac{\rho}{2}}(a^2 + b^2)^{\frac{\rho}{2}}$$

$$\times (\frac{1 + 4|\tau|^2}{v^2})^{\frac{\rho+k}{2}}(c^2 + d^2)^{\frac{-k-\rho}{2}}$$

$$+ K2^{\sigma-\frac{k}{2}}(|\tau|^2 + v^{-2})^{\sigma-\frac{k}{2}}(c^2 + d^2)^{\sigma-\frac{k}{2}}v^{-\sigma}\}.$$

Since the nonzero c, $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \Gamma(1)$, with $\Gamma(1)$ discrete, have a positive lower bound, it follows that $c^2 + d^2$ has a positive lower bound; hence

$$|\varphi_{[C_h,(0,0)]}|[\gamma,(0,0)](\tau, z)| < e^{2\pi m\frac{y^2}{v}}\{K_1(a^2 + b^2)^{\frac{\rho}{2}}(|\tau|^2 + v^{-2})^{\frac{\rho}{2}}(\frac{1 + 4|\tau|^2}{v^2})^{\frac{\rho+k}{2}}$$

$$+ K_1'(c^2 + d^2)^{\sigma-\frac{k}{2}}(|\tau|^2 + v^{-2})^{\sigma-\frac{k}{2}}v^{-\sigma}\}.$$

Note that

$$a^2 + b^2 + c^2 + d^2 = \mu(\gamma) \leq K_2\mu(A),$$

so that

$$|\varphi_{[C_h,(0,0)]}|[\gamma,(0,0)](\tau, z)| \leq e^{2\pi m\frac{y^2}{v}}\{K_3\mu(A)^{\frac{\rho}{2}}(|\tau|^2 + v^{-2})^{\frac{\rho}{2}}v^{-k-\rho}(1 + 4|\tau|^2)^{\frac{\rho+k}{2}}$$

$$+ K_3'\mu(A)^{\sigma-\frac{k}{2}}v^{-\sigma}(|\tau|^2 + v^{-2})^{\sigma-\frac{k}{2}}\}.$$

Letting $e = max(\frac{\rho}{2}, \sigma - \frac{k}{2})$, we have

$$
\begin{aligned}
|\varphi_{[C_h,(0,0)]}|[\gamma,(0,0)](\tau,z)| &\leq e^{2\pi m \frac{y^2}{v}}\{K_4\mu(A)^e(|\tau|^2 + v^{-2}) \\
&\quad \times (v^{-k-\rho}(1+4|\tau|^2)^{\rho+\frac{k}{2}} + v^{-\sigma})\} \\
&\leq e^{2\pi m \frac{y^2}{v}}\{K_4\mu(A)^e(|\tau|^2 + v^{-2})^e \\
&\quad \times (\frac{1}{2}v^{-2k-2\rho} + \frac{1}{2}(1+4|\tau|^2)^{\rho+k} + v^{-\sigma})\}.
\end{aligned}
$$

Now $\sigma \leq e + \frac{k}{2}$ and $\rho + k \leq 2e + k$, so that

$$
\begin{aligned}
|\varphi_{[C_h,(0,0)]}|[\gamma,(0,0)](\tau,z)| &\leq e^{2\pi m \frac{y^2}{v}}\{K_5\mu(A)^e(|\tau|^2 + v^{-2})^e(|\tau|^{4e+2k} + v^{-4e-2k})\} \\
&\leq e^{2\pi m \frac{y^2}{v}}\{K_6\mu(A)^e(|\tau|^{6e+2k} + v^{-6e-2k})\}.
\end{aligned}
$$

We now deal with the case in which $C_h = S^m$ for some $m \in \mathbb{Z}$. Then

$$\varphi_{[S,(0,0)]} = \varphi_0|[S,(0,0)] - \varphi_0,$$

and therefore

$$\varphi_{[C_h,(0,0)]} = \varphi_0|[C_h,(0,0)] - \varphi_0.$$

From this it follows that

$$\varphi_{[C_h,(0,0)]}|[C_{h+1}\cdots C_q,(0,0)] = \varphi_0|[C_h\cdots C_q,(0,0)] - \varphi_0|[C_{h+1}\cdots C_q,(0,0)].$$

The previous argument applies to each of the two terms on the righthand side to yield

$$|\varphi_{[C_h,(0,0)]}|[C_{h+1}\cdots C_q,(0,0)](\tau,z)| \leq e^{2\pi m \frac{y^2}{v}}\{K_7\mu(A)^e(|\tau|^{6e+2k} + v^{-6e-2k})\}.$$

The proof is completed. $\square$

Now we will use the Ford fundamental region $\mathcal{R}$. It is defined as follows:

$$\mathcal{R} = \{\tau \in \mathcal{H}|\ u < \frac{\lambda}{2} \text{ and } |c\tau + d| > 1 \text{ for all } \gamma = (\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}) \in \Gamma - \Gamma_\infty\}.$$

Then there exists $y_0 > 0$ with $iy_0 \in \mathcal{R}$. Now determine M by the condition that $[A,(\lambda,0)] \in M$ if $-\frac{\lambda}{2} \leq Re\{A(iy_0)\} < \frac{\lambda}{2}$.

Lemma 8.3. *If $[A,(\lambda,0)] = [(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}),(\lambda,0)] \in M$, chosen as indicated above, then*

$$\mu(A) \leq K'(c^2 + d^2),$$

for $K' > 0$, independent of A.

Lemma 8.4. *For $\lambda \in \mathbb{Z}$*

$$|\varphi_{[I,(\lambda,0)]}(\tau, z)| < |\lambda| K(|\tau|^\rho + v^{-\sigma}) e^{2\pi m \frac{y^2}{v}}.$$

Proof First note that $\varphi_{[I,(0,0)]} = 0$ and

$$\varphi_{[I,(0,0)]} = \varphi_{[I,(-1,0)]}|[I, (1,0)] + \varphi_{[I,(1,0)]}.$$

So $|\varphi_{I,(-1,0)]}(\tau, z)| = |\varphi_{I,(1,0)]}(\tau, z)|$. And for $\lambda > 0$ we see that

$$|\varphi_{[I,(\lambda,0)]}(\tau, z)| = |\varphi_{[I,(\lambda-1,0)]}|[I, (1,0)](\tau, z) + \varphi_{[I,(1,0)]}(\tau, z)|.$$

So by induction on λ we get the result in the case of $\lambda > 0$. And we can prove the result when $\lambda < 0$ by the same way. $\qquad\square$

Lemma 8.5. *(1) The series*

(8.2)
$$\sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^{2\pi m (\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d})}$$

converges absolutely if $k > 3$.

(2) The series

(8.3)
$$\sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} \lambda e^{2\pi m (\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d})}$$

converges absolutely if $k > 4$.

Proof (1) We consider the series

$$\sum_{x \in \mathbb{Z}} e^{-\alpha(x+\beta)^2}.$$

Note that the following estimate holds:

$$\sum_{x \in \mathbb{Z}} e^{-\alpha(x+\beta)^2} \le 1 + 2 \sum_{x \in \mathbb{N}} e^{-\alpha x^2}.$$

This is clear for $\beta \in \mathbb{Z}$. And if $\beta \notin \mathbb{Z}$, it follows from

$$-(x + \beta)^2 \le \begin{cases} -(x + [\beta] + 1)^2 & \text{for } x \le -[\beta] - 1, \\ -(x + [\beta])^2 & \text{for } x \ge -[\beta], \end{cases}$$

where $[\beta]$ is the Gauss bracket. Since $e^{-\alpha x^2} > 0$ and this is a decreasing function we get the following estimate

$$\sum_{x \in \mathbb{N}} e^{-\alpha x^2} \leq \int_0^\infty e^{-\alpha x^2} dx.$$

And if we use

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$

we get

(8.4)
$$\sum_{x \in \mathbb{Z}} e^{-\alpha(x+\beta)^2} \leq 1 + \sqrt{\frac{\pi}{\alpha}}.$$

Now we estimate the sum

$$\sum_{\lambda \in \mathbb{Z}} |e^{2\pi m(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d})}|$$

$$= |e^{2\pi m(\frac{-cz^2}{c\tau+d})}| \sum_{\lambda \in \mathbb{Z}} e^{-2\pi m(\lambda^2 \frac{v}{|c\tau+d|^2} + 2\lambda \frac{Im(z(c\bar{\tau}+d))}{|c\tau+d|^2})}$$

$$= e^{2\pi m \frac{y^2}{v}} \sum_{\lambda \in \mathbb{Z}} e^{-2\pi m \frac{v}{|c\tau+d|^2}(\lambda + \frac{Im(z(c\bar{\tau}+d))}{v})^2}.$$

If we use (8.4) then we see that the series (8.2) converges absolutely if $k > 3$.

(2) Note that if $Re(\alpha) < 0$ then we have

$$\int_{-\infty}^\infty x e^{\alpha x^2 + \beta x} dx = \frac{\sqrt{\pi}}{\sqrt{-\alpha}} \left(-\frac{\beta}{2\alpha}\right) e^{-\frac{\beta^2}{4\alpha}}$$

and

$$|Im(z(c\bar{\tau}+d))| \leq |z||c\tau+d|.$$

Then by the same argument we see that the series (8.3) converges absolutely if $k > 4$. □

Note that the series (8.2) can be written as

$$\sum_{[A,(\lambda,0)] \in \Gamma(1)_\infty^J \backslash \Gamma(1)^J} 1|_{k,m}[A, (\lambda, 0)]$$

and the series (8.3) can be written as

$$\sum_{[A,(\lambda,0)] \in \Gamma(1)_\infty^J \backslash \Gamma(1)^J} \lambda(1|_{k,m}[A, (\lambda, 0)]).$$

Proof of Theorem 5.2 Suppose $[A, (\lambda, 0)] \in M$. As before write $A = C_1 \cdots C_q$. Then we find that

$$\varphi_{[A,(0,0)]} = \varphi_{[C_1 \cdots C_q, (0,0)]}$$

$$= \varphi_{[C_1,(0,0)]} |[C_2 \cdots C_q, (0,0)] + \varphi_{[C_2,(0,0)]} |[C_3 \cdots C_q, (0,0)] + \cdots + \varphi_{[C_q,(0,0)]},$$

with $q \leq m_1 log \mu(A) + m_2$ terms on the right-hand side. By Lemma 8.2, we have

$$|\varphi_{[A,(0,0)]}(\tau, z)| \leq e^{2\pi m \frac{y^2}{v}} K^* \mu(A)^e (|\tau|^\eta + v^{-\eta}) q \leq e^{2\pi m \frac{y^2}{v}} K_1^* \mu(A)^{e+1} (|\tau|^\eta + v^{-\eta}),$$

where $\eta = 6e + 2k$ and we have used $q \leq m_1 log \mu(A) + m_2 \leq m_3 \mu(A)$. Lemma 8.3 yields

$$|\varphi_{[A,(0,0)]}(\tau, z)| \leq e^{2\pi m \frac{y^2}{v}} K_2^* (c^2 + d^2)^{e+1} (|\tau|^\eta + v^{-\eta}),$$

and, by Lemma 8.1,

(8.5) $$|\varphi_{[A,(0,0)]}(\tau, z)| \leq e^{2\pi m \frac{y^2}{v}} K_2^* |c\tau + d|^{2e+2} (\frac{1 + 4|\tau|^2}{v^2})^{e+1} (|\tau|^\eta + v^{-\eta}).$$

Note that

$$|\varphi_{[A,(\lambda,0)]}(\tau, z)| \leq |\varphi_{[A,(0,0)]}| [I, (\lambda, 0)](\tau, z)| + |\varphi_{[I,(\lambda,0)]}(\tau, z)|.$$

Hence, by (8.5) and Lemma 8.4

$$\begin{aligned} |\varphi_{[A,(\lambda,0)]}(\tau, z)| &\leq e^{2\pi m \frac{y^2}{v}} K_2^* |c\tau + d|^{2e+2} (\frac{1 + 4|\tau|^2}{v^2})^{e+1} (|\tau|^\eta + v^{-\eta}) \\ &\quad + |\lambda| K (|\tau|^\rho + v^{-\sigma}) e^{2\pi m \frac{y^2}{v}}. \end{aligned}$$

To prove the convergence of the series $\Phi(\tau, z)$ we need to estimate the absolute value of the general term of the series. This is

$$\begin{aligned} &|\varphi_\gamma(\tau, z)(1|_{\omega,k,m}[A, (\lambda, 0)])(\tau, z)| \\ &< e^{2\pi m \frac{y^2}{v}} K_2^* |c\tau + d|^{2e+2} (\frac{1 + 4|\tau|^2}{v^2})^{e+1} (|\tau|^\eta + v^{-\eta}) |(1|_{\omega,k,m}[A, (\lambda, 0)])(\tau, z)| \\ &\quad + |\lambda| K (|\tau|^\rho + v^{-\sigma}) e^{2\pi m \frac{y^2}{v}} |(1|_{\omega,k,m}[A, (\lambda, 0)])(\tau, z)| \\ &= e^{2\pi m \frac{y^2}{v}} K_2^* (\frac{1 + 4|\tau|^2}{v^2})^{e+1} (|\tau|^\eta + v^{-\eta}) |(1|_{\omega,k-2e-2,m}[A, (\lambda, 0)])(\tau, z)| \\ &\quad + |\lambda| K (|\tau|^\rho + v^{-\sigma}) e^{2\pi m \frac{y^2}{v}} |(1|_{\omega,k,m}[A, (\lambda, 0)])(\tau, z)|, \end{aligned}$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. By Lemma 8.5 we know that the series $\Phi(\tau, z)$ converges if $k > 2e + 5$ and $k > 4$. $\square$

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